Exercise set 3 - Kinematics

Reminders

Simplified notation of sines and cosines

To simplify the notation, we use:

- $\sin(\theta) = s$
- $\cos(\theta) = c$
- $\sin(\theta_1) = s_1$
- $\cos(\theta_1) = c_1$
- $\sin(\theta_2) = s_2$
- $\cos(\theta_2) = c_2$
- $\cos(\theta_1 + \theta_2) = c_{1+2}$
- $\sin(\theta_1 + \theta_2) = s_{1+2}$

Rotation and translation matrices

Recall that:

- $\mathbf{R}(\theta) = \begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ describes the rotation of θ around the origin (in 2D)
- $\mathbf{R_x}(\theta)$ describes the rotation of θ around the axis x
- $\mathbf{R_y}(\theta)$ describes the rotation of θ around the axis y
- $\mathbf{R_z}(\theta)$ describes the rotation of θ around the axis z
- $\mathbf{t} = \begin{pmatrix} t_x \\ t_y \end{pmatrix}$ describes the translation vector $\mathbf{t}$

Sequence of transformations

The sequence $a \rightarrow b \rightarrow c$ describes the transformation a followed by the transformation b followed by the transformation c.

Quaternions

The quaternion $\mathbf{Q}$:

$$\mathbf{Q} = \begin{pmatrix} \lambda_0 \\ \lambda_x \\ \lambda_y \\ \lambda_z \end{pmatrix} = \begin{pmatrix} \lambda_0 \\ \boldsymbol{\lambda} \end{pmatrix} = \lambda_0 + i\lambda_x + j\lambda_y + k\lambda_z$$

describes a rotation with a rotation axis $\boldsymbol{\lambda}$ and a rotation angle θ such that $\lambda_0 = \cos(\theta/2)$ and $\boldsymbol{\lambda} = \sin(\theta/2)[x,y,z]^T$ with $||[x,y,z]||=1$.

Scalar product:

$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}|cos(\vec{u},\vec{v})$

Cross product:

Let $\vec{u} = \begin{pmatrix} u_x \\ u_y \\ u_z \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} v_x \\ v_y \\ v_z \end{pmatrix}$. Then $\vec{u} \times \vec{v} = \begin{pmatrix} u_y v_z - v_y u_z \\ v_x u_z - u_x v_z \\ u_x v_y - v_x u_y \end{pmatrix}$

Exercise 1

Find the matrices of pure rotation around the three axes of the Cartesian system:

1. $\mathbf{R_x}$ around x.
2. $\mathbf{R_y}$ around y.
3. $\mathbf{R_z}$ around z.

Exercise 2

Consider the following two sequences of operations:

$$\mathbf{R_z}(90°) \rightarrow \mathbf{R_y}(90°)$$
$$\mathbf{R_y}(90°) \rightarrow \mathbf{R_z}(90°)$$

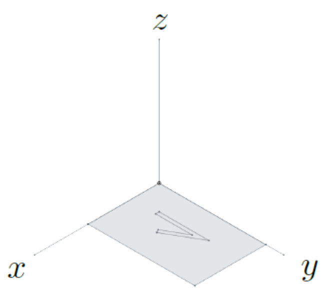
Give the rotation matrices corresponding to these sequences. Are they equivalent?

Exercise 3

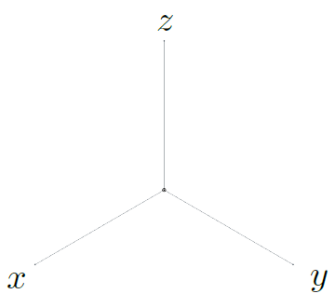
Consider the two sequences from the previous exercise:

$$\mathbf{R_z}(90°) \rightarrow \mathbf{R_y}(90°)$$
$$\mathbf{R_y}(90°) \rightarrow \mathbf{R_z}(90°)$$

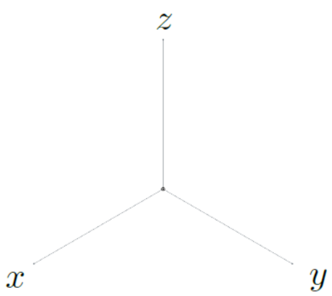
For each of the two sequences, determine graphically by iteration the result of the sequence using an object oriented in a Cartesian coordinate system in isometric projection, as in the figure below ('b' and 'c' are to be completed by you):



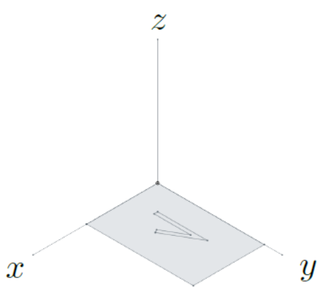
(a) object in initial position



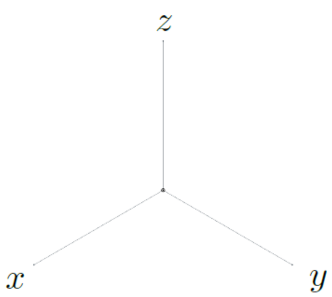
(b) object after $\mathbf{R_z}(90°)$, to be completed



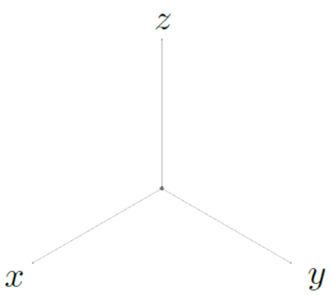
(c) object after $\mathbf{R_z}(90°) \rightarrow \mathbf{R_y}(90°)$, to be completed



(a) object in initial position



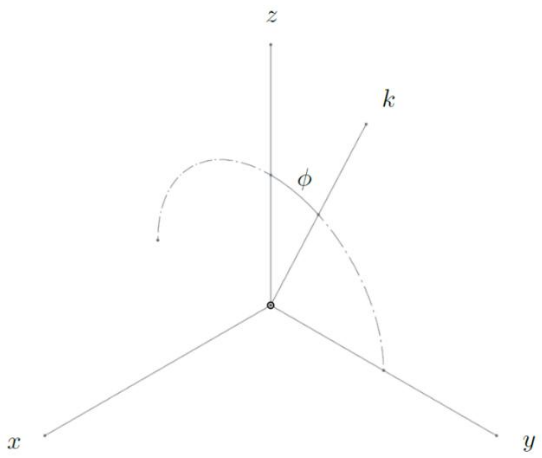
(b) object after $\mathbf{R_y}(90°)$, to be completed



(c) object after $\mathbf{R_y}(90°) \rightarrow \mathbf{R_z}(90°)$, to be completed

Exercise 4

Find the matrix of direction cosines for a rotation with an angle θ around an axis k, which is in the yz plane and which is inclined by an angle ϕ with respect to the axis z, i.e find the rotation matrix corresponding to this transformation. **Hint:** use a sequence of basic rotations ($\mathbf{R_x}$, $\mathbf{R_y}$ or $\mathbf{R_z}$).



Exercise 5

Consider the two sequences of exercises 1 and 2:

$$\mathbf{R_z}(90°) \rightarrow \mathbf{R_y}(90°)$$

$$\mathbf{R_y}(90°) \rightarrow \mathbf{R_z}(90°)$$

For each of these sequences:

1. Determine the resulting corresponding quaternion.
2. Deduce:
   (a) the corresponding angles of rotation.
   (b) the corresponding unit axes of rotation.

Exercise 6

Consider an object with vertices A, B, C, transformed in such a way that its vertices are found at A', B', C'; the vectors giving the coordinates of the points are **a**, **b**, **c**, **a'**, **b'** and **c'**:

$$\mathbf{a} = \begin{pmatrix}1\\0\\0\end{pmatrix} \mathbf{b} = \begin{pmatrix}0\\0\\0\end{pmatrix} \mathbf{c} = \begin{pmatrix}0\\1\\0\end{pmatrix}$$

$$\mathbf{a}' = \begin{pmatrix}1\\0\\1\end{pmatrix} \mathbf{b}' = \begin{pmatrix}1\\-1\\1\end{pmatrix} \mathbf{c}' = \begin{pmatrix}0\\-1\\1\end{pmatrix}$$

1. Find the rotation (angle and axis) and the translation (offset and axis) corresponding to the transformation. **Hint:** use a drawing.
2. Deduce the corresponding homogeneous transformation matrix.